# Schwitzer Times

Answers to 2021 Puzzles

**Andromeda Delivered.** The orbital periods look a bit hairy, but it's their ratios that determine planetary alignments, and the problem is easier than it looks. Apollonis (period **7 months**) orbits exactly **8 times** as fast as Cephisso (period **56 months**), and Borysthenis (period **14/10 years**) orbits **10/3 times** as fast as Cephisso (period **14/3 years**). Now planets in near-circular orbits move at near-constant speeds, so alignments, if they reoccur, must occur at regular intervals (a fact known to the ancient inhabitants of Earth through the eclipse cycle). We don't have a starting position for the planets, but assuming the inhabitants have live-tested their transporter lift, we can assume a full alignment has occurred at least once.

It will be easiest to work in terms of orbital positions, so using **x** for the portion of the orbit of Cephisso (the slowest moving planet), we see that Apollonis will next be aligned after it has moved a bit more than half an orbit, so when **$x + ½ = 8x$** which gives **$x = 1/14$**.

Similarly for Borysthenis, it will next be aligned with Cephisso when **$x + ½ = (10/3)x$** which gives **$x = 3/14$**.

Alignments of each pair of planets must repeat after these intervals, so full alignment of all three planets happens next at their lowest common multiple i.e. after **3/14** of an orbit of Cephisso. How long is that in time? **3/14** of **14/3** years is exactly **12 months**, so **Santa delivers presents every year**.

You probably already knew that from childhood!!!

**The Postman's Solution.** The key to the puzzle is that for each friend the sum of the house numbers lower than theirs is the same as the sum of the house numbers higher than theirs. If you worked out that Charlie lived at **6 Chapel Close** but didn't get the other two, then congratulate yourself, your arithmetic is accurate and your problem solving is (moderately) tenacious! As a check: 1 + 2 + 3 + 4 + 5 = **15** = 7 + 8.

Unfortunately rather more trial and error is needed for the other streets as we don't know how many houses there are, so a bit of insight will come in handy. Recall that the sum of the numbers up to **n** is **½n(n+1)**, a result easily remembered by imagining the sum laid out as a triangle of coins, and seeing that two such triangles can fit together to make an **n** by **(n+1)** rectangle. Then for a friend living in a street of **s** houses, the sums below and above his house h, which we're told are the same, are

$$\tfrac{1}{2}(h-1)h = \tfrac{1}{2}s(s+1) - \tfrac{1}{2}h(h+1)$$

which simplifies to

$$2h^2 = s(s+1).$$

Welcome to Diophantine equations! – a grand term simply meaning solutions must be whole numbers. The name honours Diophantus of Alexandria, whose *Arithmetica* books, written in the third century A.D., covered them. Seventeen centuries is quite old even for a maths text book, so some copies were lost, and one copy was ruined by Fermat scribbling speculative stuff in the margin. In any event, other mathematicians have managed to pick up what Diophantus started.

In our Diophantine equation, we need two successive numbers **s** and **s+1** whose product is twice a square $h^2$. Equating the prime factors on each side, on the left the prime factors are paired with an additional 2, so on the right (as two successive numbers cannot share a divisor) either **s** is itself a square and **s+1** twice a square, or the other way round. This now makes the numbers manageable, so it's easy to find solutions. For example using Excel, list the **numbers 1 to 22** in column 1, put **twice col1 squared** in column 2, and in columns 3 and 4 put the square roots of **col2 + 1** and **col2 –1** respectively, then spot the whole numbers! You "only" need 22 rows, as $2x23^2$ is already more than three digits! You'll see solutions alternate between columns 3 and 4, and that there are only three solutions with house numbers not more than three digits, so given what we're told about the lengths of the streets, the other two friends can only live at **35 Beech Street** and **204 Acacia Avenue**.

If you need to check: for Beech Street 1 + 2 + ... + 34 = **595** = 36 + ... + 49, and for Acacia Avenue 1 + 2 + ... + 203 = **20706** = 205 + ... + 288.

However, the puzzle gives more. One of the anecdotes about Srinivasa Ramanujan, student of Trinity, Cambridge, usually considered the greatest "natural" mathematician, was that given this problem with the number of houses in the street restricted to between 50 and 500, he quickly (while cooking a meal!) came up with house 204 in a street of 288. On being asked how he solved it so fast replied "by continued fractions". How could Ramanujan have done that?

His first thought after arriving at $\mathbf{2h^2 = s(s+1)}$ might well have been spotting

$$\mathbf{h}\ \textit{tends towards}\ \frac{1}{\sqrt{2}}\mathbf{s}$$

for large **s**, that is, in longer streets the house must be a bit more than two-thirds the way along the street (which feels about right). Now Ramanujan was an expert with continued fractions, now rarely taught, and a "well-known" continued fraction expansion for **sqrt(2)** is

$$\sqrt{2} = 1 + \frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\ldots$$

with successive truncations giving progressively better approximations to **sqrt(2)**, a topic sometimes called Diophantine Approximations. That these truncations give exact solutions to $\mathbf{2h^2 = s(s+1)}$ seems an incredible (if not beautiful) result, but for Ramanujan was a simple fact among a lot of deep mathematical insight. In fact he only needed to evaluate the third term

$$1 + \frac{1}{2+}\frac{1}{2+}\frac{1}{2} = 1 + \frac{1}{2+}\frac{2}{5} = 1 + \frac{5}{12} = \frac{17}{12} = 1.41666\ldots$$

(which either he knew or did in his head) to get the fraction **17/12** and so the required solution of house **12x17 = 204** in a street of $\mathbf{2 \times 12^2 = 288}$ houses.

On the other hand, Ramanujan with his intimate knowledge of numbers might just have remembered that $\mathbf{2 \times 12^2 + 1 = 289 = 17^2}$.

Another way of finding solutions $\mathbf{(u_n, v_n)}$ is by induction – which in fact uses exactly the same arithmetic as when you simplify successive stages of the continued fractions above.

Let's start with a known even-numbered solution $\mathbf{(u_2, v_2) = (2, 3)}$ which gives Chapel Close, that is house **2x3 = 6** in a street of $\mathbf{2 \times 2^2 = 8}$ houses. (In terms of the original problem, the house is $\mathbf{h = u_n v_n}$ in a street of $\mathbf{s = 2u_n^2}$ houses). We know that $\mathbf{v_n^2 - 1 = 2u_n^2}$ for **n=2**. Now construct the next trial solution as

$$\boxed{\mathbf{(u_{n+1}, v_{n+1}) = (u_n + v_n, 2u_n + v_n).}}$$

We see that

$$\begin{aligned}
\mathbf{v_{n+1}^2 + 1} &\mathbf{= (2u_n + v_n)^2 = 4u_n^2 + 4u_nv_n + v_n^2 + 1} \\
&\mathbf{= 2u_n^2 + 4u_nv_n + v_n^2 + (2u_n^2 + 1)} \\
&\mathbf{= 2u_n^2 + 4u_nv_n + 2v_n^2} \\
&\mathbf{= 2u_{n+1}^2}
\end{aligned}$$

which shows $\mathbf{(u_{n+1}, v_{n+1})}$ is an odd solution (with **+1** rather than **–1** in the formula).

Constructing the same way the next trial even solution, we see

$$\begin{aligned} v_{n+2}^2 - 1 &= (2u_{n+1} + v_{n+1})^2 \\ &= 2u_{n+1}^2 + 4u_{n+1}v_{n+1} + v_{n+1}^2 + (2u_{n+1}^2 - 1) \\ &= 2u_{n+1}^2 + 4u_{n+1}v_{n+1} + 2v_{n+1}^2 \\ &= 2u_{n+2}^2 \end{aligned}$$

so **$(u_{n+2}, v_{n+2})$** is indeed an even solution, and so by induction every **$(u_n, v_n)$** constructed in this way is a solution.

Finally, unlike problems with primes, we see that the solutions to this problem occur absolutely regularly (geometrically speaking), each successive solution being larger by a factor of about **6** (the exact multiple tends towards **3 + 2sqrt(2) = 5.828...** ).

We hope this simple problem has provided suitable inspiration and in some cases refreshed parts not often reached!